

Blow-up rate for a semi-linear accretive wave equation

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Abstract. In this paper we consider the semi-linear wave equation: $u_{tt} - \Delta u = u_t|u_t|^{p-1}$ in \mathbb{R}^N . We provide an associated energy. With this energy we give the blow-up rate for blowing up solutions in the case of bounded below energy.

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1 Introduction

We consider the following semi-linear wave equation:

$$\begin{cases} u_{tt} - \Delta u = u_t|u_t|^{p-1} & t \in [0, T), x \in \mathbb{R}^N \\ (u(x, 0), u_t(x, 0)) = (u_0, u_1) \in Y_{loc, u}(\mathbb{R}^N), \end{cases} \quad (1)$$

where $Y_{loc, u}$ is either $Y_{loc, u}^{2,2} := H_{loc, u}^2(\mathbb{R}^N) \times H_{loc, u}^1(\mathbb{R}^N)$ or $Y_{loc, u}^{2,\infty} := (W_{loc, u}^{2,\infty} \cap H_{loc, u}^2(\mathbb{R}^N)) \times (L_{loc, u}^\infty \cap H_{loc, u}^1(\mathbb{R}^N))$, with

$$L_{loc, u}^2(\mathbb{R}^N) = \left\{ v: \mathbb{R}^N \rightarrow \mathbb{R}; \|v\|_{L_{loc, u}^2} := \sup_{x_0 \in \mathbb{R}^N} \int_{|x-x_0| \leq 1} |v(x)|^2 dx < \infty \right\}$$

and

$$H_{loc, u}^1(\mathbb{R}^N) := \{v \in L_{loc, u}^2(\mathbb{R}^N); |\nabla v| \in L_{loc, u}^2(\mathbb{R}^N)\},$$

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and corresponding definitions for $L_{loc,u}^\infty$ and $W_{loc,u}^{k,\infty}$. We assume in addition that

$$\left\{ \begin{array}{l} (u_0, u_1) \in Y_{loc,u}^{2,2} \text{ and } \left| \begin{array}{l} p \in \mathbb{N} \cap \left(1, \frac{N}{N-2}\right) \text{ if } N \geq 3 \\ \text{or} \\ p \in \{2, 3, \dots\} \text{ if } N \leq 2. \end{array} \right. \\ (u_0, u_1) \in Y_{loc,u}^{2,\infty}, N \leq 3, \text{ and } p \in \mathbb{R} \cap (1, +\infty) \end{array} \right. \quad (2)$$

All of this is due to the local existence result (see for instance Faour, Fino and Jazar [5]).

A very rich literature has been done on the semi-linear equation

$$u_{tt} - \Delta u = au_t |u_t|^{p-1} + bu|u|^{q-1} \quad (3)$$

with a, b, p and q are real numbers, $p, q \geq 1$. When $a \leq 0$ and $b = 0$ then the damping term $au_t |u_t|^{p-1}$ assume global existence in time for arbitrary data (see, for instance, Harraux and Zuazua [7] and Kopackova [9]). When $a \leq 0, b > 0$ and $p > q$ then one can cite, for instance, Levine [10] and Georgiev and Todorova [4], that show the existence of global solutions (in time) under negative energy condition. When $a \leq 0, b > 0$ and $q > p$, or when $a \leq 0, b > 0$ and $p = 1$ then one can cite [4] and Messaoudi [15] where they show finite time blowing up solutions under sufficiently large negative energy of the initial condition.

The first to consider the case $a > 0$ was Haraux [6] (with $b = 0$ on bounded domain), who construct blowing up solutions for arbitrary small initial data. See also Jazar and Kiwan [8] and the references therein for the same equation (1) on bounded domain. We refer to Levine, Park and Serrin [11] and the references therein for the whole space-case \mathbb{R}^N . Finally, we refer to Haraux [6], Souplet [17, 18] and Jazar and Souplet [3] concerning the ODE case.

Unlike previous work where the considered question was to provide conditions ensuring finite time blowup for the solution, recent interesting work has been done aiming at understanding the behavior of blowing up solutions in $H_{loc,u}^1 \times L_{loc,u}^2$ -norm. This was the aim of the paper of Antonini and Merle [2] and also the series of papers of Merle and Zaag [12, 13, 14] where they was concerned by the blow-up rate for (3) in the case $a = 0$ and $b = 1$. They showed that the blow-up rate is that of the associated ODE ($u'' = u^p$) for $1 < p \leq 1 + \frac{4}{N-1}$, and in [14] they study the growth rate near the blowup surface.

In this paper we consider the case $a = 1$ and $b = 0$.

For the rest of the paper, and following [2, 12, 13, 14], we consider solutions u of (1) that blow-up in finite time $T > 0$ in the space $Y_{loc,u}(\mathbb{R}^N)$. Our aim is to study the blow-up behavior of $u(t)$ as $t \uparrow T$. We compare the growth of u_t and k , the solution of the simplest associated ODE: $k_{tt} =$

$k_t|k_t|^{p-1}$. Nevertheless, the presence of the force term $u_t|u_t|^{p-1}$ makes the work more complicated. To remedy this difficulty, and inspired by the work of Rivera and Fatori [16], we rewrite (1) as

$$\begin{cases} u_{tt} - \int_0^t \Delta u_t(\tau) d\tau - \Delta u_0 = u_t |u_t|^{p-1}, & t \in [0, T), x \in \mathbb{R}^N, \\ (u(x, 0), u_t(x, 0)) = (u_0(x), u_1(x)) \in Y_{loc, u}. \end{cases} \quad (4)$$

Then, putting

$$v(x, t) = u_t(x, t) \quad (5)$$

in (4), we obtain the following integro differential PDE

$$\begin{cases} v_t - \int_0^t \Delta v(\tau) d\tau - \Delta u_0(x) = v |v|^{p-1} & t \in [0, T), x \in \mathbb{R}^N, \\ v(x, 0) = u_1(x) =: v_0. \end{cases} \quad (6)$$

Now, we introduce (see [1, 2]) the following change of variables. For $a \in \mathbb{R}^N$ and $T' > 0$, with $\beta := \frac{1}{p-1}$,

$$z = x - a, \quad s = -\log(T' - t), \quad v(t, x) = (T' - t)^{-\beta} \theta_{T', a}(s, z) \quad (7)$$

and

$$(T')^{\beta+1} u_0(x) =: \theta_{a,00} \quad (T')^\beta v(x, 0) =: \theta(s_0, z) =: \theta_{a,0}(z)$$

where $s_0 := -\log T'$. We then see that the function $\theta_{T', a}$ (we write θ for simplicity) satisfies for all $s \geq -\log T'$ (and $s < -\log(T' - T)$ if $T' > T$) and all $z \in \mathbb{R}^N$

$$g(s)\theta_s + \beta g(s)\theta - \int_{s_0}^s g_2(\tau) \Delta \theta(\tau) d\tau - g(s_0) \Delta \theta_{00} = g(s) |\theta|^{p-1} \theta \quad (8)$$

where $g(s) := e^{(\beta+1)s}$ and $g_2(s) := e^{(\beta-1)s}$. Denote by $h(s) := e^{-(\beta+1)s}$ and $h_2(s) := e^{-(\beta-1)s}$.

In the new set of variables (s, z) , the behavior of u_t as $t \uparrow T$ is equivalent to the behavior of θ as $s \rightarrow \infty$.

In Section 2 we define an associated energy to equation (8) that is decreasing (see Proposition 1).

Our main result in this paper is:

Theorem 1 (*Bounds on θ*)

Assume that (u_0, u_1) and p satisfy (2). If u is a blowing-up solution at time $T > 0$ of (1) and θ is defined as in (7) and satisfies

$$C \leq E[\theta](s) \quad (9)$$

for some constant C and for all $s \geq s_0$, then there exists $K > 0$ that depends on N , p and bounds on T and the initial data in $Y_{loc,u}$ such that

$$\sup_{\substack{s > s_0 \\ a \in \mathbb{R}^N}} \left[e^{-2s} \|h_2 \star \theta(s, \cdot)\|_{H^1(B)}^2 + \|\theta(s, \cdot)\|_{L^2(B)}^2 \right] < K,$$

where B is the unit ball of \mathbb{R}^N and $h_2 \star \theta(s, z) := \int_{s_0}^s h_2(s - s') \theta(s', z) ds'$.

This can be translated in terms of u :

Theorem 2 (Bounds on blowing-up solutions of (1))

Assume that (u_0, u_1) and p satisfy (2). If u is a blowing-up solution at time $T > 0$ of equation (1) and θ is defined as in (7) satisfying (9), then there exists a positive constant C , that depends on N , p and bounds on T and the initial data in $Y_{loc,u}$, such that for all $t \in [T(1 - e^{-1}), T)$, and all $a \in \mathbb{R}^N$:

$$(T - t)^{2\beta} \left[\|u\|_{H^2(B_a)}^2 + \|u_t(t)\|_{H^1(B_a)}^2 \right] \leq C,$$

where B_a is the unit ball centered at a .

In Section 3 we provide the proof of Theorem 1. In the last section we improve the regularity of the solution by providing a control on the L^r -norm of θ and $e^{-s}[h_2 \star \theta]$ for $1 \leq r \leq \frac{2N}{N-2}$.

Acknowledgment. This work is strongly inspired by the series of papers of Merle and Zaag [12, 13, 14].

2 The associated energy

In this section we define a weighted energy associated to the equation (8) as follows:

$$\begin{aligned} E(s) &:= \beta/2 \int_B g(s) \rho^\alpha \theta(s)^2 dz - \frac{1}{p+1} \int_B g(s) \rho^\alpha |\theta(s)|^{p+1} dz \\ &\quad - \frac{1}{8} \int_{s_0}^s \int_B \rho^\alpha g_2(\tau) [4|\nabla \theta(\tau) - \nabla \theta(s)|^2 - |\nabla \theta(s)|^2] dz d\tau \\ &\quad - \alpha \int_{s_0}^s \int_B g_2(\tau) [N\rho - 2(\alpha - 1)z^2] \rho^{\alpha-2} [|\theta(s) - \theta(\tau)|^2 - |\theta(s)|^2] dz d\tau \\ &\quad - \alpha \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} [|e^{-2\tau} z \nabla \theta(s) - \theta(\tau)|^2 - |e^{-2\tau} z \nabla \theta(s)|^2] dz d\tau \\ &\quad + \frac{1}{2} g(s_0) \int_B \rho^\alpha [|\nabla \theta(s) + \nabla \theta_{00}|^2 - |\nabla \theta(s)|^2] dz \\ &\quad + \alpha g(s_0) \int_B \rho^{\alpha-1} [|\theta(s) - z \nabla \theta_{00}|^2 - |\theta(s)|^2] dz, \end{aligned}$$

where B denotes the unit ball, α is any number satisfying $\alpha > \max\{\beta(\beta + 1)/2, 1 + 2\beta, 2\}$, and $\rho(z) := 1 - |z|^2$.

In this section we prove the following

Proposition 1 *The energy $s \mapsto E(s)$ is a decreasing function for $s \geq s_0$. Moreover, we have*

$$\begin{aligned}
E(s+1) - E(s) &= -\frac{\beta+1}{p+1} \int_s^{s+1} \int_B g(s) \rho^\alpha |\theta(s')|^{p+1} dz ds' \\
&\quad - \int_s^{s+1} \int_B g(s) \rho^\alpha \theta_s^2(s') dz ds' \\
&\quad - [\alpha - \beta(\beta+1)/2] \int_s^{s+1} g(s') \int_B \rho^\alpha \theta^2(s') dz ds' \\
&\quad - \alpha \int_s^{s+1} \int_B g(s') \rho^{\alpha-1} |z|^2 |\theta(s')|^2 dz ds' \\
&\quad - \int_s^{s+1} \int_B g_2(s') \rho^\alpha |\nabla \theta(s')|^2 dz ds'.
\end{aligned} \tag{10}$$

Proof: In order to calculate the derivative of E , multiply the equation (8) by $\rho^\alpha \theta_s$ and integrate over $B := \{|z| \leq 1\}$. Then we get

$$\begin{aligned}
\int_B g(s) \rho^\alpha [\beta \theta \theta_s - \theta |\theta|^{p-1} \theta_s] dz - \int_{s_0}^s \int_B \rho^\alpha g_2(\tau) \theta_s(s) \Delta \theta(\tau) dz d\tau - B' \\
= - \int_B g(s) \rho^\alpha \theta_s^2 dz.
\end{aligned} \tag{11}$$

with

$$\begin{aligned}
B' &:= g(s_0) \int_B \rho^\alpha \Delta \theta_{00}(s_0, z) \theta_s(s, z) dz \\
&= -g(s_0) \int_B \rho^\alpha \nabla \theta_s \nabla \theta_{00} dz + 2\alpha g(s_0) \int_B \rho^{\alpha-1} z \theta_s \nabla \theta_{00} dz \\
&= -(B_1 + B_2).
\end{aligned}$$

We have

$$\begin{aligned}
B_1 &= g(s_0) \int_B \rho^\alpha \nabla \theta_s \nabla \theta_{00} dz \\
&= \frac{1}{2} g(s_0) \frac{d}{ds} \int_B \rho^\alpha [|\nabla \theta(s) + \nabla \theta_{00}|^2 - |\nabla \theta(s)|^2] dz,
\end{aligned}$$

and

$$\begin{aligned}
B_2 &= -2\alpha g(s_0) \int_B \rho^{\alpha-1} z \theta_s \nabla \theta_{00} \\
&= \alpha g(s_0) \frac{d}{ds} \int_B \rho^{\alpha-1} [|\theta(s) - z \nabla \theta_{00}|^2 - |\theta(s)|^2] dz.
\end{aligned}$$

Using Green's formula, we write the term

$$-\int_{s_0}^s \int_B \rho^\alpha g_2(\tau) \theta_s(s) \Delta \theta(\tau) dz d\tau = I_1 + I_2$$

where

$$I_1 := \int_{s_0}^s \int_B g_2(\tau) \rho^\alpha \nabla \theta_s(s) \nabla \theta(\tau) dz d\tau$$

and

$$I_2 := -2\alpha \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} \theta_s(s) z \nabla \theta(\tau) dz d\tau.$$

For I_1 :

$$\begin{aligned} I_1 &= -\frac{1}{2} \frac{d}{ds} \int_{s_0}^s \int_B \rho^\alpha g_2(\tau) |2\nabla \theta(\tau) - \frac{1}{2} \nabla \theta(s)|^2 dz d\tau \\ &\quad + \frac{1}{8} \frac{d}{ds} \left[\int_{s_0}^s g_2(\tau) d\tau \int_B \rho^\alpha |\nabla \theta(s)|^2 dz \right] \\ &\quad + g_2(s) \int_B \rho^\alpha |\nabla \theta(s)|^2 dz. \end{aligned}$$

Remainder I_2 :

$$\begin{aligned} I_2 &= -2\alpha \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} \theta_s(s) z \nabla \theta(\tau) dz d\tau \\ &= 2\alpha \int_{s_0}^s \int_B g_2(\tau) \nabla [z \rho^{\alpha-1} \theta_s(s)] \theta(\tau) dz d\tau \\ &= 2N\alpha \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} \theta_s(s) \theta(\tau) dz d\tau \\ &\quad - 4\alpha(\alpha-1) \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-2} z^2 \theta_s(s) \theta(\tau) dz d\tau \\ &\quad + 2\alpha \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} z \nabla \theta_s(s) \theta(\tau) dz d\tau \\ &= A_1 + A_2 + A_3, \end{aligned}$$

with

$$\begin{aligned} A_1 &:= 2N\alpha \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} \theta_s(s) \theta(\tau) dz d\tau \\ &= -N\alpha \frac{d}{ds} \left[\int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} |\theta(s) - \theta(\tau)|^2 dz d\tau \right] \\ &\quad + N\alpha \frac{d}{ds} \left[\int_{s_0}^s g_2(\tau) d\tau \int_B \rho^{\alpha-1} |\theta(s)|^2 dz \right] \\ &\quad - N\alpha g_2(s) \int_B \rho^{\alpha-1} |\theta(s)|^2 dz, \end{aligned}$$

$$\begin{aligned}
A_2 &:= -4\alpha(\alpha-1) \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-2} z^2 \theta_s(s) \theta(\tau) dz d\tau \\
&= 2\alpha(\alpha-1) \frac{d}{ds} \left[\int_{s_0}^s \int_B g_2(\tau) z^2 \rho^{\alpha-2} |\theta(s) - \theta(\tau)|^2 dz d\tau \right] \\
&\quad - 2\alpha(\alpha-1) \frac{d}{ds} \left[\int_{s_0}^s g_2(\tau) d\tau \int_B z^2 \rho^{\alpha-2} |\theta(s)|^2 dz \right] \\
&\quad + 2\alpha(\alpha-1) g_2(s) \int_B z^2 \rho^{\alpha-2} |\theta(s)|^2 dz,
\end{aligned}$$

$$\begin{aligned}
A_3 &:= 2\alpha \int_{s_0}^s \int_B g_2(\tau) \rho^{\alpha-1} z \nabla \theta_s(s) \theta(\tau) dz d\tau \\
&:= 2\alpha \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} e^{-2\tau} z \nabla(\theta_s(s)) \theta(\tau) dz d\tau \\
&= -\alpha \frac{d}{ds} \left[\int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} |e^{-2\tau} z \nabla \theta(s) - \theta(\tau)|^2 dz d\tau \right] \\
&\quad + \alpha \frac{d}{ds} \left[\int_{s_0}^s g_4(\tau) d\tau \int_B z^2 \rho^{\alpha-1} |\nabla \theta(s)|^2 dz \right] \\
&\quad + \alpha \frac{d}{ds} \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} |\theta(\tau)|^2 dz d\tau - 2\alpha g_2(s) \int_B \rho^{\alpha-1} z \nabla \theta(s) \theta(s) dz \\
&= -\alpha \frac{d}{ds} \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} |e^{-2\tau} z \nabla \theta(s) - \theta(\tau)|^2 dz d\tau \\
&\quad + \alpha \frac{d}{ds} \left[\int_{s_0}^s g_4(\tau) d\tau \int_B z^2 \rho^{\alpha-1} |\nabla \theta(s)|^2 dz \right] \\
&\quad + \alpha g(s) \int_B \rho^{\alpha-1} |\theta(s)|^2 dz - \alpha g_2(s) \int_B \rho^{\alpha-1} z \nabla \theta^2(s) dz \\
&= -\alpha \frac{d}{ds} \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} |e^{-2\tau} z \nabla \theta(s) - \theta(\tau)|^2 dz d\tau \\
&\quad + \alpha \frac{d}{ds} \left[\int_{s_0}^s g_4(\tau) d\tau \int_B z^2 \rho^{\alpha-1} |\nabla \theta(s)|^2 dz \right] \\
&\quad + \alpha g(s) \int_B \rho^{\alpha-1} |\theta(s)|^2 dz + \alpha g_2(s) \int_B \nabla(\rho^{\alpha-1} z) \theta^2(s) dz \\
&= -\alpha \frac{d}{ds} \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} |e^{-2\tau} z \nabla \theta(s) - \theta(\tau)|^2 dz d\tau \\
&\quad + \alpha \frac{d}{ds} \left[\int_{s_0}^s g_4(\tau) d\tau \int_B z^2 \rho^{\alpha-1} |\nabla \theta(s)|^2 dz \right] \\
&\quad + \alpha g(s) \int_B \rho^{\alpha-1} |\theta(s)|^2 dz + \alpha N g_2(s) \int_B \rho^{\alpha-1} \theta^2(s) dz \\
&\quad - 2\alpha(\alpha-1) g_2(s) \int_B \rho^{\alpha-2} |z|^2 \theta^2(s) dz.
\end{aligned}$$

Then

$$\begin{aligned}
I_2 &= -\alpha \frac{d}{ds} \int_{s_0}^s \int_B g_2(\tau) [N\rho - 2(\alpha - 1)z^2] \rho^{\alpha-2} |\theta(s) - \theta(\tau)|^2 dz \\
&\quad + \alpha \frac{d}{ds} \int_{s_0}^s g_2(\tau) d\tau \int_B [N\rho - 2(\alpha - 1)z^2] \rho^{\alpha-2} |\theta(s)|^2 dz \\
&\quad - \alpha \frac{d}{ds} \int_{s_0}^s \int_B g(\tau) \rho^{\alpha-1} |e^{-2\tau} z \nabla \theta(s) - \theta(\tau)|^2 dz d\tau \\
&\quad + \alpha \frac{d}{ds} \int_{s_0}^s g_4(\tau) d\tau \int_B z^2 \rho^{\alpha-1} |\nabla \theta(s)|^2 dz \\
&\quad + \alpha g(s) \int_B \rho^{\alpha-1} |\theta(s)|^2 dz.
\end{aligned}$$

Putting B_1 , B_2 , I_1 and I_2 into (11) we finally get

$$\begin{aligned}
\frac{d}{ds} E(s) &= -\frac{\beta+1}{p+1} \int_B g(s) \rho^\alpha |\theta(s)|^{p+1} dz - \int_B g(s) \rho^\alpha \theta_s^2(s) dz \\
&\quad - [\alpha - \beta(\beta+1)/2] \int_B g(s) \rho^\alpha \theta^2(s) dz - \alpha \int_B g(s) \rho^{\alpha-1} |z|^2 |\theta(s)|^2 \\
&\quad - \int_B g_2(s) \rho^\alpha |\nabla \theta(s)|^2.
\end{aligned} \tag{12}$$

which terminates the proof of the lemma. \square

3 Bounds on θ : Proof of Theorem 1

We start by the following corollary of Proposition 1

Corollary 1 (Bounds on E and θ) *For all $s \geq s_0$ we have*

$$C \leq E[\theta(s)] \leq E[\theta(s_0)] =: C_0, \tag{13}$$

$$\int_s^{s+1} \int_B g(s') \rho^\alpha (\theta_s^2 + |\theta|^{p+1} + \theta^2) dy ds' + \int_s^{s+1} \int_B g_2(s') \rho^\alpha |\nabla \theta|^2 \leq C, \tag{14}$$

$$\int_s^{s+1} \int_B \rho^\alpha (\theta_s^2 + |\theta|^2 + |\theta|^{p+1} + |\nabla \theta|^2) dy ds' \leq C, \tag{15}$$

$$\int_s^{s+1} \int_{B_{1/2}} (\theta_s^2 + |\theta|^2 + |\theta|^{p+1} + |\nabla \theta|^2) dy ds' \leq C, \tag{16}$$

where C depends only on bounds on T , and the initial data of (1) in $Y_{loc,u}$.

Proof: Inequalities (13) and (14) follow directly from Proposition 1. Inequality (15) follows from (14) writing

$$\begin{aligned} \int_s^{s+1} \int_B \rho^\alpha (\theta_s^2 + |\theta|^{p+1} + |\nabla \theta|^2 + \theta^2) dy ds' &\leq \min(h(s_0), h_2(s_0)) \times \\ &\times \left[\int_s^{s+1} \int_B g(s') \rho^\alpha (\theta_s^2 + |\theta|^{p+1} + \theta^2) dy ds' + \int_s^{s+1} \int_B g_2(s') \rho^\alpha |\nabla \theta|^2 \right] \leq C. \end{aligned}$$

Similarly, since $\rho^\alpha \geq 3/4$ over $B_{1/2}$, inequality (16) follows from (15). \square

The proof of Theorem 1 will be done in the following three propositions:

Proposition 2 (Control of θ in $L^2_{loc,u}$) For all $s \geq s_0 + 1$ and all $a \in \mathbb{R}^N$ we have

$$\int_B \theta_a^2 dz \leq C. \quad (17)$$

Proposition 3 (Control of $e^{-s}[h_2 \star \nabla \theta]$ in $L^2_{loc,u}$) For all $s \geq s_0 + 1$ and all $a \in \mathbb{R}^N$ we have

$$e^{-2s} \int_B |h_2 \star \nabla \theta_a(s, z)|^2 dz \leq C. \quad (18)$$

Proposition 4 (Control of $e^{-s}[h_2 \star \theta]$ in $L^2_{loc,u}$) For all $s \geq s_0 + 1$ and all $a \in \mathbb{R}^N$ we have

$$e^{-2s} \int_B [h_2 \star \theta_a]^2 dz \leq C.$$

Strategy of the proof: Following [13] and by a covering technique, we start showing that we can insert ρ^α inside the integral \int_B , then, using mean value theorem, we bound \int_B by $\int_s^{s+1} \int_B$. We terminate by straightforward (but tricky) calculations using inequalities of Corollary 1.

Proof of proposition 2: 1. Let $a_0 := a_0(s)$ be such that

$$\int_B \rho^\alpha \theta_{a_0}^2(s, z) dz \geq \frac{1}{2} \sup_{a \in \mathbb{R}^N} \int_B \rho^\alpha \theta_a^2(s, z) dz.$$

We have:

Lemma 1 For all $s \geq s_0 + 1$ and for any $a \in \mathbb{R}^N$, we have

$$\int_B \theta_a^2(s, z) dz \leq C \int_B \rho^\alpha \theta_{a_0}^2(s, z) dz. \quad (19)$$

Proof of Lemma 1: Using the definition (7) of θ and the fact that $\rho \geq 3/4$ over $B_{1/2}$ we have

$$\begin{aligned} \int_{B_{\frac{1}{2}}} \theta_a^2(z_0 + z, s) dz &= \int_{B_{\frac{1}{2}}} \theta_{a+z_0}^2(z, s) dz \\ &\leq C \int_{B_{\frac{1}{2}}} \rho^\alpha \theta_{a+z_0}^2(z, s) dz \\ &\leq C \sup_{a \in \mathbb{R}^N} \int_B \rho^\alpha \theta_a^2 dz \leq 2C \int_B \rho^\alpha \theta_{a_0}^2 dz, \end{aligned}$$

uniformly with respect to $z_0 \in B$. Now since we can cover the ball B with $k(N)$ balls of radius $1/2$, this proves (19). \square

2. Remains to prove that

$$\int_B \rho^\alpha \theta_{a_0}^2(s, z) dz \leq C.$$

Using the mean value theorem and (15), there exists $\tau \in [s, s+1]$ such that

$$\int_B \rho^\alpha \theta_{a_0}^2(\tau, z) dz = \int_s^{s+1} \int_B \rho^\alpha \theta_{a_0}^2(s', z) dz ds' \leq C.$$

Now

$$\begin{aligned} \int_B \rho^\alpha \theta_{a_0}^2(s, z) dz &= \int_B \rho^\alpha \theta_{a_0}^2(\tau, z) dz - \int_s^\tau \int_B \rho^\alpha \frac{\partial}{\partial s} [\theta_{a_0}^2](s', z) dz ds' \\ &\leq C - \int_s^\tau \int_B \rho^\alpha 2\theta_{a_0}(\theta_{a_0})_s(s', z) dz ds' \\ &\leq C + \int_s^\tau \int_B \rho^\alpha [\theta_{a_0}^2 + (\theta_{a_0})_s^2] dz ds' \\ &\leq C + \int_s^{s+1} \int_B \rho^\alpha [\theta_{a_0}^2 + (\theta_{a_0})_s^2] dz ds' \\ &\leq 3C \quad (\text{by (15)}). \end{aligned}$$

This ends the proof of Proposition 2. \square

Proof of Proposition 3: 1. For $s \geq s_0 + 1$ let $a_1 = a_1(s)$ be such that

$$e^{-2s} \int_B \rho^\alpha \left[\int_{s_0}^s h_2(s-s') \nabla \theta_{a_1} ds' \right]^2 dz \geq \frac{1}{2} \sup_{a \in \mathbb{R}^N} e^{-2s} \int_B \rho^\alpha \left[\int_{s_0}^s h_2(s-s') \nabla \theta_a ds' \right]^2 dz.$$

We need the following:

Lemma 2 For all $s \geq s_0 + 1$ and for any $a \in \mathbb{R}^N$, we have

$$e^{-2s} \int_B [h_2 \star \nabla \theta_a(s, z)]^2 dz \leq C e^{-2s} \int_B \rho^\alpha [h_2 \star \nabla \theta_{a_1}]^2 dz. \quad (20)$$

Proof of Lemma 2: Using the definition (7) of θ and the fact that $\rho \geq 3/4$ over $B_{1/2}$ we have

$$\begin{aligned}
e^{-2s} \int_{B_{\frac{1}{2}}} \left[\int_{s_0}^s h_2(s-s') \nabla \theta_a(z_0 + z, s') ds' \right]^2 dz \\
&= e^{-2s} \int_{B_{\frac{1}{2}}} \left[\int_{s_0}^s h_2(s-s') \nabla \theta_{a+z_0}(z, s') ds' \right]^2 dz \\
&\leq C e^{-2s} \int_{B_{\frac{1}{2}}} \rho^\alpha \left[\int_{s_0}^s h_2(s-s') \nabla \theta_{a+z_0}(z, s') ds' \right]^2 dz \\
&\leq C \sup_{a \in \mathbb{R}^N} e^{-2s} \int_B \rho^\alpha \left[\int_{s_0}^s h_2(s-s') \nabla \theta_a(z, s') ds' \right]^2 dz \\
&\leq C e^{-2s} \int_B \rho^\alpha \left[\int_{s_0}^s h_2(s-s') \nabla \theta_{a_1}(z, s') ds' \right]^2 dz,
\end{aligned}$$

uniformly with respect to $z_0 \in B$. Now since we can cover the ball B with $k(N)$ balls of radius $1/2$, this proves (20). \square

2. Now we will prove that

$$\int_s^{s+1} e^{-2s'} \int_B \rho^\alpha [h_2 \star \nabla \theta_{a_1}]^2(s', z) ds' dz \leq C. \quad (21)$$

By integration by parts we have

$$\begin{aligned}
\int_B \rho^\alpha \Delta \theta(s', z) \theta(s, z) dz &= - \int_B \rho^\alpha \nabla \theta(s', z) \nabla \theta(s, z) dz \\
&\quad + 2\alpha \int_B \rho^{\alpha-1} \theta(s, z) z \cdot \nabla \theta(s', z) dz.
\end{aligned} \quad (22)$$

Thus

$$\begin{aligned}
\int_s^{s+1} e^{-2s'} \int_B \rho^\alpha \theta [h_2 \star \Delta \theta] dz ds' &= - \int_s^{s+1} e^{-2s'} \int_B \rho^\alpha [h_2 \star \nabla \theta] \cdot \nabla \theta dz ds' \\
&\quad + 2\alpha \int_s^{s+1} e^{-2s'} \int_B \rho^{\alpha-1} z \cdot [h_2 \star \nabla \theta] \theta dy ds'.
\end{aligned} \quad (23)$$

Now, since

$$\frac{\partial}{\partial s} [e^{-s} (h_2 \star f)] = e^{-s} [f - \beta (h_2 \star f)],$$

so, for $s_1 < s_2$ we have

$$\frac{1}{2} \left[e^{-2s'} |h_2 \star \nabla \theta|^2 \right]_{s_1}^{s_2} = \int_{s_1}^{s_2} e^{-2s'} [h_2 \star \nabla \theta] \cdot \nabla \theta ds' - \beta \int_{s_1}^{s_2} e^{-2s'} |h_2 \star \nabla \theta|^2 ds'. \quad (24)$$

Multiplying equation (8) by $\rho^\alpha \theta_{a_1}$ and then integrating over $[s, s+1] \times B$ we get (using (22), (23) and (24))

$$\begin{aligned} & \frac{1}{2} \int_B \rho^\alpha \left[e^{-2s'} |h_2 \star \nabla \theta_{a_1}|^2 \right]_s^{s+1} dz + \beta \int_s^{s+1} \int_B \rho^\alpha e^{-2s'} |h_2 \star \nabla \theta_{a_1}|^2 dz ds' \\ & \quad - 2\alpha \int_s^{s+1} \int_B \rho^{\alpha-1} e^{-2s'} z \cdot [h_2 \star \nabla \theta_{a_1}] \theta_{a_1} dz ds'. \\ & = - \int_s^{s+1} \int_B \rho^\alpha \theta_{a_1} [(\theta_{a_1})_s + \beta \theta_{a_1} - h(s-s_0) \Delta \theta_{00} - |\theta_{a_1}|^{p+1}] dz ds'. \end{aligned}$$

Using the inequality $\pm ab \leq \gamma^{-1} a^2 + \frac{\gamma}{4} b^2$, we have

$$\int_s^{s+1} \int_B \rho^{\alpha-1} z e^{-2s'} [h_2 \star \nabla \theta] \theta \leq \gamma^{-1} \int_s^{s+1} \int_B \rho^\alpha e^{-2s'} |h_2 \star \nabla \theta|^2 + \frac{\gamma}{4} \int_s^{s+1} \int_B \rho^{\alpha-2} |z|^2 \theta^2,$$

where $\gamma = 4\frac{\alpha}{\beta}$. Then, using (15) and proposition 2, we get

$$\frac{\beta}{2} \int_s^{s+1} \int_B \rho^\alpha e^{-2s'} |h_2 \star \nabla \theta_{a_1}|^2 dz ds' + \frac{1}{2} \int_B \left[e^{-2s'} |h_2 \star \nabla w_{a_1}|^2 \right]_s^{s+1} dy \leq C. \quad (25)$$

This can be written as

$$y'(s) + ay(s) \leq b,$$

where a and b are positive constants and

$$y(s) := \int_s^{s+1} \int_B \rho^\alpha e^{-2s'} |h_2 \star \nabla \theta_{a_1}|^2 dz ds'.$$

This directly gives (21).

3. Remains to prove that, for all $s \geq s_0 + 1$, we have

$$\int_B \rho^\alpha e^{-2s} |h_2 \star \nabla \theta_{a_1}|^2(s, z) dz \leq C. \quad (26)$$

Using the mean value theorem and (15), there exists $\tau \in [s, s+1]$ such that

$$\int_B \rho^\alpha e^{-2\tau} [h_2 \star \nabla \theta_{a_1}]^2(\tau, z) dz = \int_s^{s+1} \int_B \rho^\alpha e^{-2s'} [h_2 \star \nabla \theta_{a_1}]^2(s', z) dz ds' \leq C.$$

Then

$$\begin{aligned} \int_B \rho^\alpha e^{-2s} [h_2 \star \nabla \theta_{a_1}]^2(s, z) dz &= \int_B \rho^\alpha e^{-2\tau} [h_2 \star \nabla \theta_{a_1}]^2(\tau, z) dz \\ &\quad + \int_s^\tau \int_B \rho^\alpha \frac{\partial}{\partial s} (e^{-2s'} [h_2 \star \nabla \theta_{a_1}]^2(s', z)) ds' dz \\ &\leq C + 2 \int_s^\tau \int_B \rho^\alpha e^{-2s'} [h_2 \star \nabla \theta_{a_1}] [\nabla \theta_{a_1} - \beta h_2 \star \nabla \theta_{a_1}] ds' dz \end{aligned}$$

$$\begin{aligned}
&\leq C + 2 \int_s^\tau \int_B \rho^\alpha e^{-2s'} (h_2 \star \nabla \theta_{a_1}) \nabla \theta_{a_1} \\
&\quad - 2 \int_s^\tau \int_B \rho^\alpha \beta e^{-2s'} |h_2 \star \nabla \theta_{a_1}|^2 \} ds' dz \\
&\leq C + C_1 \int_s^{s+1} \int_B \rho^\alpha e^{-2s'} |h_2 \star \nabla \theta_{a_1}|^2 \\
&\quad + C_2 \int_s^{s+1} \int_B \rho^\alpha \nabla \theta_{a_1}^2 ds' dz \\
&\leq C' \quad (\text{by (15)}).
\end{aligned}$$

This ends the proof of Proposition 3. \square

Proof of proposition 4: The proof is similar to the previous one. \square

4 Improvement of the regularity to L^r , $1 \leq r \leq \frac{2N}{N-2}$

We terminate with an improvement of the control on θ and $e^{-s}[h_2 \star \theta]$ we obtained in Propositions 2 and 4. In fact, using Sobolev's embedding Theorem and the covering technique used in Propositions 2, 3 and 4 we can show the following:

Proposition 5 (Control of θ and $e^{-s}[h_2 \star \theta]$ in $L^r(B)$ for $1 \leq r \leq \frac{2N}{N-2}$)
Let $1 \leq r \leq \frac{2N}{N-2}$. For all $s \geq s_0 + 1$ and all $a \in \mathbb{R}^N$ we have

$$e^{-rs} \int_B |(h_2 \star \theta_a(s, z))|^r dz \leq C. \quad (27)$$

If, in addition, $r \leq \frac{2N}{N-1}$ then

$$\int_B |\theta_a(s, z)|^r dz \leq C. \quad (28)$$

Proof of Proposition 5: The inequality (27) is direct using propositions 3, 4 and Sobolev's injection Theorem: $H^1(B) \hookrightarrow L^r(B)$.

For the inequality (28) and following the proof of Proposition 2, let $a_3 := a_3(s)$ be such that

$$\int_B \rho^\alpha \theta_{a_3}^r(s, z) dz \geq \frac{1}{2} \sup_{a \in \mathbb{R}^N} \int_B \rho^\alpha \theta_a^r(s, z) dz,$$

where $\theta_{a_3}^r$ stand for $|\theta_{a_3}|^r$. Similarly, we get:

$$\int_B \theta_{a_3}^r(s, z) dz \leq C \int_B \rho^\alpha \theta_{a_3}^r(s, z) dz. \quad (29)$$

Using the mean value theorem and (15), there exists $\tau \in [s, s+1]$ such that

$$\int_B \rho^\alpha \theta_{a_3}^r(\tau, z) dz = \int_s^{s+1} \int_B \rho^\alpha \theta_{a_3}^r(s', z) dz ds' \leq C.$$

Now, using Sobolev's embedding theorem $H^1((s, s+1) \times B) \hookrightarrow L^{2(r-1)}((s, s+1) \times B)$, we get

$$\begin{aligned} \int_B \rho^\alpha \theta_{a_3}^r(s, z) dz &= \int_B \rho^\alpha \theta_{a_3}^r(\tau, z) dz + \int_s^\tau \int_B \rho^\alpha \frac{\partial}{\partial s} \theta_{a_3}^r(s', z) ds' dz \\ &\leq C + \int_s^\tau \int_B \rho^\alpha r |\theta_{a_3}|^{r-1} |(\theta_{a_3})_s|(s', z) ds' dz \\ &\leq C + \frac{1}{2} \int_s^\tau \int_B \rho^\alpha [|\theta_{a_3}|^{2(r-1)} + (\theta_{a_3})_s^2] ds' dz \\ &\leq C + C \left[\int_s^{s+1} \int_B \rho^\alpha [\theta_{a_3}^2 + |\nabla \theta_{a_3}|^2] ds' dz \right]^{r-1} \\ &\quad + \frac{1}{2} \int_s^{s+1} \int_B \rho^\alpha (\theta_{a_3})_s^2 ds' dz \\ &\leq C \quad (\text{by (15)}). \end{aligned}$$

This ends the proof of Proposition 5. □

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